The tontine puzzle

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Abstract

Under both normative and descriptive decision theory, tontines and tontine-related retirement income products have proven their superiority to classical annuity products. In the present paper, we show that benefits of a properly designed tontine dominate the benefits of an equally priced annuity as the pool size tends to infinity, leading individuals to prefer tontines with a sufficiently large pool size to annuities under utility preferences which are increasing and continuous in consumption. Such preferences include but are not limtied to cumulative prospect theory and generalized expected utility preferences, which we use as examples to illustrate our theoretical findings. Our results present an interesting puzzle which we call "tontine puzzle", raising the question why the development of the tontine market is still in its infancy in practice.

Keywords: Retirement planning, behavioral economics, risk sharing, annuities and tontines

JEL: G22, H55, H23, I31, J32

1 Introduction

Tontines are financial products, where a part of the returns is obtained via the inheritance of wealth from non-surviving participants. Tontines date back to the 17th century (Weir (1989)) when they were staged as public financial instruments to raise funds. As many countries are faced with the challenges caused by an ageing society, OECD (2020) points out the importance of longevity risk-sharing schemes between retirees and/or pension providers in the retirement landscape. Modern tontines and tontine-linked products which allow a better longevity risk sharing between the policyholders and retirement products providers have inevitably gained increasing popularity both in practice and among academic research. Despite its upward moving trend, the development of the tontine market is still rather in its infancy in practice. Table 1 provides an overview of several limited tontines offered in practice. For example, Nuovalo offers a so-called LifePools platform to retirement benefit providers. Using this platform, providers are enabled to include mortality risk sharing schemes such as tontines in their portfolio of retirement plans. As a second example, GuardPath Longevity Solutions offers a “GuardPath Modern Tontine”, “Hybrid Tontine Series” and “GuardPath Managed Decumulation” to beneficiaries.

Name	Provider	Country	URL to official website
Tontine Pension	Tontine Trust	Ireland	`https://tontine.com/`
La Tontine	Le Conservateur	France	`https://www.conservateur.fr/`
LifePools	Nuovalo Ltd.	U.S.A.	`https://www.nuovalo.com/`
GuardPath Longevity Solutions	Guardian Capital	Canada	`https://www.guardiancapital.com`

Table 1: Some tontines offered in practice.

In many recent academic papers comparing the attractiveness of tontines (or tontine-linked products) with annuities, tontines have proven their superiority to annuity products (cf. Stamos (2008), Hanewald et al. (2013), Milevsky and Salisbury (2015), Chen et al. (2019, 2020b, 2021), Bommier and Schernberg (2021) and Weinert and Gründl (2021)). However, in all of these articles, a specific (descriptive or normative) utility or value function is applied, making the superiority of tontines only applicable with certain restrictions. Further, the comparison is typically conducted between optimal (i.e. utility/value-maximizing) tontines and optimal annuities (see below for detailed elaborations on the existing literature). In this paper, we focus on practically implementable versions of annuities and tontines, in particular constant annuities and natural tontines as introduced in Milevsky and Salisbury (2015). In a natural tontine, retirement benefits are chosen such that individuals receive a constant retirement income for life if mortality evolves as expected. Constant annuities and natural tontines are easy to communicate and can be considered a compromise between theoretical optimality and practical suitability. With these products, we show that the retirement benefits of a tontine dominate those of an equally priced annuity under mild assumptions as the pool size tends to infinity, making the benefits of the tontine preferable under any utility preferences preserving the monotonicity of consumption if the pool size of the tontine is sufficiently large. The wide general theoretical support for tontines (or tontine-like products) and their limited widespread in practice present a rather interesting puzzle, which we name as "Tontine Puzzle".

Assuming fair pricing and life-cycle utility preferences with temporal risk neutrality (economic agents are only risk averse about consumption, but not about their lifetime)[1] and concave utility preferences for consumption, annuities are the preferable retirement product from a retiree's perspective (cf. Yaari (1965) and Milevsky and Salisbury (2015)). However, under various realistic assumptions, many recent studies find at least partial tontinization combined with partial annuitization to be preferred to full annuitization under various scenarios:

- Assuming concave utility preferences for consumption, Stamos (2008), Hanewald et al. (2013), Milevsky and Salisbury (2015), Bernhardt and Donnelly (2019), Chen et al. (2019, 2020b, 2021) and Chen and Rach (2022) verify this argument under temporal risk neutrality if *realistic risk loadings* are charged for annuities.
- Chen et al. (2020a) find that individuals who are subject to *subjective* mortality beliefs might perceive tontines as more attractive than annuities. This is particularly the case for individuals underestimating their peers' remaining lifetimes compared to the insurer.
- Assuming concave utility preferences for consumption, Bommier and Schernberg (2021) show that individuals exhibiting *temporal risk aversion* prefer at least a fraction of their wealth to depend on realized survival probabilities (i.e. prefer investing in tontine-linked products), even in perfect markets with fair insurance.

[1]Under temporal risk aversion, economic agents are not only risk averse about consumption, but also about their lifetime. In contrast, under temporal risk neutrality, agents are only risk averse about consumption, but not their lifetime.

- Finally, Weinert and Gründl (2021) consider pensioners with *multi cumulative prospect theory* preferences and find this result to be valid for fair tontines and annuities if individuals' wealth falls into a certain range.

In total, both normative and descriptive approaches suggest that tontines can at least be a beneficial supplement to annuities. However, all the above literature deals with optimal tontines under very specific utility assumptions, which limits their results in generality and applicability to the retirement market: First, in practice, the retirement products provided by insurance companies might be optimal (utility- or value optimizing) for some customers, but certainly not for all. Insurers typically provide a pre-selected set of benefits which are easy to communicate with most of customers. Second, even if insurers were willing to offer all types of utility-maximizing payoff structures (which would be a tremendous effort) to policyholders, they probably behave sub-rationally with their decisions, e.g. due to psychological bias or computational limitations (see, e.g., Hu and Scott (2007) and Liu et al. (2022)). Third and finally, the use of one particular (normative or descriptive) utility preference limits the results in their generality, as not all individuals exhibit only one and the same type of preferences. Hence, in the current study, we therefore focus on practically easy-to-communicate and easily implementable natural tontines and constant annuities. The rather simple structure of these retirement benefits enables us to derive more general conclusions on the superiority of tontines than under a specific utility maximization approach.

We find that tontine benefits dominate equally priced annuity benefits under mild assumptions as the pool size tends to infinity, implying that the superiority of tontines with a sufficiently large pool is valid for any utility preference which is increasing and continuous in consumption. Such preferences include normative approaches like generalized life-cycle utility preferences allowing for temporal risk aversion (from now on referred to as EUT), as long as utility functions are increasing and continuous (but not necessarily concave), as well as descriptive approaches such as cumulative prospect theory (CPT). The main assumptions for this result are a sufficiently large tontine pool and the presence of risk loadings or distorted survival probabilities underestimating best-estimate survival probabilities. These main driving factors for the superiority of tontines over annuities are consistent with the existing literature on optimal tontines.

Our results put forward the demand for tontines and tontine-linked products. This can also partly resolve the annuity puzzle, one of most complex puzzles in modern economy. It describes the discrepancy between the theoretically optimal demand for annuities (see e.g. Yaari (1965), Peijnenburg et al. (2016)) and the empirically observed demand for annuities (see e.g. Inkmann et al. (2010)). According to the pioneering article Yaari (1965), individuals should fully annuitize all wealth unless there is a bequest motive. Since it has been pointed out in Chen and Rach (2022) that a bequest motive does not change the preference order of annuities and tontines under expected utility, we disregard bequest motives in this article. In the past years, economists across the world have extensively studied the annuity puzzle and delivered a variety of new insights (e.g. Hu and Scott (2007), Agnew et al. (2008), Beshears et al. (2014) and Salisbury and Nenkov (2016)). Our paper contributes to this literature by providing a rationale why a high percentage of annuitization is not optimal.

The remainder of this article is organized as follows. In Section 2, we present the essential notation used throughout the article. In Sections 3 and 4, we present our main results and demonstrate them theoretically and numerically in two well-known settings. Section 5 concludes the paper and is followed by a technical appendix.

2 Notations

We consider an individual at retirement age x (in whole years) at time $t = 0$ whose remaining lifetime is given by T_x. This individual is endowed with an initial wealth $W_0 > 0$ which is used to buy either an immediate annuity or a tontine, i.e. the present value of future consumption may not exceed the initial wealth. The maximum age that any individual can reach is denoted by ω (e.g. 120).

In an annuity product, an insurer pools many independent retirees and promises each of them a life-long, regular stream of retirement benefits. Hence, an annuity delivers the payoff

$$b_c(k) := \mathbb{1}_{\{T_x > k\}} c(k), \ k = 0, 1, \dots, \omega - x \tag{1}$$

to a single retiree, where $c(k)$ is a deterministic payout function specified at contract initiation, and $\mathbb{1}_{\{B\}}$ is an indicator function that is equal to one if event B occurs and zero otherwise.

In a tontine product (Milevsky and Salisbury (2015)), the insurer issues a retirement payment scheme at time 0 to $n \in \mathbb{N}_0$ homogeneous contract holders of the same age x and of the same gender. At the beginning of each year $k \in \{0, 1, \dots, \omega - x\}$, each surviving individual obtains a deterministic payment $d(k)$ (specified at time 0) and an additional mortality credit given by $\frac{(n-N(k))d(k)}{N(k)}$, where $N(k)$ denotes the number of survivors at time k. The resulting annual payoff at any time $k \in \{0, 1, \ldots, \omega - x\}$ is given by

$$b_d(k) = \begin{cases} \mathbb{1}_{\{T_x > k\}} \frac{n}{N(k)} d(k), & N(k) > 0, \\ 0, & N(k) = 0. \end{cases} \tag{2}$$

The payoff (2) breaks down to an *annuity* payoff if the pool consists of only one individual, i.e. $n = 1$. Throughout this article, we assume the future lifetimes of individuals to be independent.

Throughout this article, we assume that individuals and retirement product providers rely on (potentially) different estimates of future survival probabilities. To calculate the expected present value of future benefits, the insurer relies on the risk-neutral pricing approach (cf. e.g. Cairns et al. (2006)) to account for prudence: It chooses a risk-neutral probability measure $\mathbb{Q}$ which typically differs from the real-world measure $\mathbb{P}$ in such a way that $\mathbb{Q}(T_x > t) > \mathbb{P}(T_x > t)$, i.e. survival probabilities under the risk-neutral measure systematically exceed best-estimate survival probabilities. For a number of whole years k, we use $s_x(k) := \mathbb{P}(T_x > k)$ to denote the best-estimate k-year survival probability under $\mathbb{P}$ and $s_x^{\mathbb{Q}}(k) := \mathbb{Q}(T_x > k)$ to denote the k-year survival probability of a currently x-year old under the risk-neutral measure $\mathbb{Q}$.

The single premiums of retirement plans are then given by the expected present value of future benefits under the risk-neutral measure $\mathbb{Q}$. For detailed calculations of the premiums, see e.g. Chen et al. (2019). The premium of the annuity the insurer charges is given as follows:

$$P_0^c = \mathbb{E}_{\mathbb{Q}} \left[\sum_{k=0}^{\omega-x} v(0,k) c(k) \mathbb{1}_{\{T_x > k\}} \right] = \sum_{k=0}^{\omega-x} v(0,k) s_x^{\mathbb{Q}}(k)\, c(k),$$

where $v(0,k)$ is a deterministic discount function from time k to time 0. In case of a constant annuity, i.e. $c(t) \equiv c$, an individual with an initial wealth level W_0 obtains an annuity payment $c = W_0 / \sum_{k=0}^{\omega-x} v(0,k) s_x^{\mathbb{Q}}(k)$, satisfying $P_0^c = W_0$.

For the premium of the tontine, we rely on the fact that $(N(t) - 1 \mid T_x > t) \sim \mathrm{Bin}(n-1, s_x^{\mathbb{Q}}(t))$ under $\mathbb{Q}$, given the independence of the future lifetimes of the tontine participants. Therefore, we obtain the premium as (cf. Chen et al. (2019)):

$$\begin{aligned} P_0^d &= \mathbb{E}_{\mathbb{Q}}\left[\sum_{k=0}^{\omega-x} v(0,k)\frac{nd(k)}{N(k)}\mathbb{1}_{\{T_x>k\}}\right] \\ &= \sum_{k=0}^{\omega-x} v(0,k)\left(1-\left(1-s_x^{\mathbb{Q}}(k)\right)^n\right) d(k). \end{aligned}$$

In this paper, we will mainly consider the so-called natural tontine introduced by Milevsky and Salisbury (2015). With a natural tontine, the deterministic payoff $d(k)$ is determined such that the retirement benefits to a single individual remain constant over time if mortality evolves as expected (under the insurer's perspective). In other words, we choose the payoff $d(k)$ as $d(k) := \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{T_x>k\}}\right] d_0 = s_x^{\mathbb{Q}}(k) d_0$, where d_0 is a constant making the budget constraint binding. The natural tontine and constant annuity can be considered a compromise between theoretical optimality and practical suitability. For a detailed analysis of utility-maximizing annuity and tontine payoffs, we refer interested readers e.g. to Stamos (2008), Hanewald et al. (2013), Milevsky and Salisbury (2015), Bernhardt and Donnelly (2019) and Chen et al. (2019, 2020b).

To illustrate the impact of the pricing measure $\mathbb{Q}$ on the premiums, consider the Gompertz mortality model which is parameterized by the modal age at death m and the dispersion coefficient b (see Gompertz (1825)). The mortality rate in this model is given by

$$\mu_{x+t} = \frac{1}{b} e^{\frac{x+t-m}{b}}. \tag{3}$$

Note that the mortality rate is decreasing in the modal age at death m. To achieve a prudent risk-neutral measure $\mathbb{Q}$, we therefore assume that the insurer chooses a modal age $m^{\mathbb{Q}} > m$, because this choice results in $s_x^{\mathbb{Q}}(k) > s_x(k)$ for all (x,k) if $b^{\mathbb{Q}} = b$. For example, consider a 65-year old individual and the parameters $m = 88.721$ and $b = 10$ taken from Milevsky and Salisbury (2015) along with a constant risk-free interest rate of 1% p.a.. Then, we can choose $m^{\mathbb{Q}}$ such that the proportional risk loading of a constant annuity is 4%. We obtain $m^{\mathbb{Q}} = 89.885$. For a natural tontine, we get a proportional risk loading of approximately 0.0056%.

Individuals' subjective survival probabilities may as well differ from best-estimate survival probabilities, where underestimations tend to be more frequent, but overestimations may occur as well (see e.g. O'Brien et al. (2005), Greenwald and Associates (2012), Elder (2013), Wu et al. (2015)). This steam of literature also suggests that individuals' estimates for themselves differ compared to their beliefs about others' life expectancy (see e.g. O'Brien et al. (2005), Greenwald and Associates (2012)). To capture this aspect, we allow individuals to have different survival probabilities for themselves and their peers. In the following, let $\hat{s}_x(k)$ be the subjective k-year survival probability that a single agent estimates for an x-year-old peer. Let $\bar{s}_x(k)$ be the subjective survival probability which agents assign to themselves. Hence, we assume that individuals may not be fully rational and build their own subjective estimations regarding their longevity.

3 Main result and its consequences

We start by proving the central property of the tontine payoff.

Theorem 3.1 (**Benefits in the limit**). *Assume that survival probabilities under the risk-neutral measure exceed the subjective survival probabilities that an agent estimates for an x-year-old peer at all times, i.e.*

$$\hat{s}_x(k) < s_x^{\mathbb{Q}}(k) \text{ for all } k \in \{1, \dots, \omega - x\}. \tag{4}$$

Then, as the pool size n tends to ***infinity****, the benefits of a natural tontine with initial value W_0 exceed the benefits of a constant annuity with initial value W_0 at all times.*

Proof: See Appendix A.1.

Remark 3.2. *The results in Theorem 3.1 are rather general in their formulation and include a variety of special cases which we want to explain in detail below.*

- *First, if there are no subjective probabilities (in particular for others), i.e. $s_x(k) = \hat{s}_x(k) = \bar{s}_x(k)$, and $s_x(k) < s_x^Q(k)$, then the subjective probabilities coincide with best-estimate probabilities. In this case, the condition stated in inequality* (4) *is fulfilled naturally.*

- *If there are no risk loadings, the insurer relies on best-estimate probabilities for pricing, i.e. $s_x^{\mathbb{Q}}(k) = s_x(k)$. Then, the condition stated in* (4) *translates into agents' underestimating the survival probabilities of their peers and thus subjectively overestimating the tontine payoff.*[2]

[2] A similar result has already been found by Chen et al. (2020a) for more specific utility preferences.

- *Not all individuals will underestimate their peers' remaining lifetimes compared to the insurer, i.e. there are multiple situations in which Assumption* (4) *is not fulfilled. First, if an individual's subjective beliefs coincide with the insurer's pricing measure, i.e.* $\hat{s}_x(k) = s_x^{\mathbb{Q}}(k)$ *, then both retirement plans will deliver the same benefits in the limit, but, for finite pool sizes, the annuity will typically be preferred (see Milevsky and Salisbury (2015)). A special case of this situation is the following: If there are no subjective probabilities regarding others and no risk loadings, i.e.* $s_x(k) = s_x^{\mathbb{Q}}(k) = \bar{s}_x(k)$ *, we are in a fair pricing framework (see also Milevsky and Salisbury (2015) and Chen et al. (2020b)). This case, although analytically convenient, is, however, rather unrealistic, because insurers typically charge risk loadings from policyholders for annuities. Second, if individuals overestimate their peers' remaining lifetimes compared to the insurer, i.e.* $\hat{s}_x(k) > s_x^{\mathbb{Q}}(k)$ *, the tontine payoff will appear lower and therefore less attractive.*

Based on the result in Theorem 3.1, we can conclude the following.

Corollary 3.3 (Utility preferences). *Assume condition* (4)*. Then, for any utility preference that is continuous and increasing in consumption, and for any combination of a constant annuity and a natural tontine (except full annuitization), there exists a pool size* n_0 *such that an individual prefers the combination to an equally priced constant annuity and an equally priced natural tontine to the combination, if the pool size in the tontine is at least* n_0 *.*

Proof: See Appendix A.2.

The result of Corollary 3.3 might hint that the benefits of a natural tontine stochastically dominate the benefits of a constant annuity. However, when comparing a random benefit X to a constant benefit c, we can only have first- or second-order stochastic dominance between c and X if $P(X \geq c) = 1$, which is clearly not the case if we compare the tontine to the annuity benefit, i.e. if we set $X = nd(k)/N(k)$. Hence, Corollary 3.3 does not imply a first- or second-order stochastic dominance between annuities and natural tontines.

Examples for the utility preferences described in Corollary 3.3 include but are not limited to life-cycle utility preferences as used in Yaari (1965) or Milevsky and Salisbury (2015), generalized life-cycle utility preferences as used e.g. in Bommier (2006) and Bommier et al. (2011), but also descriptive models like cumulative prospect theory as used inTversky and Kahneman (1992) and Hu and Scott (2007) (assuming that the underlying utility functions are continuous and increasing, which they are in all the named references). In the following section, we consider two examples of such preferences.

4 Examples

4.1 Cumulative prospect theory

CPT was originally introduced by Tversky and Kahneman (1992). It relies on a value function V which is concave above a reference point Γ and convex below this reference point, accounting for the concept of loss aversion. Additionally, for the calculation of the CPT value, real world probabilities are transformed to subjective probabilities.

To evaluate retirement plans under CPT, following Hu and Scott (2007), we evaluate the total discounted retirement benefits at each time k compared to the total initial wealth W_0 used as reference level. Thus, individuals make a profit if they live long enough for the retirement benefits to exceed their initial investment and they suffer a loss if they die before a sufficient retirement income is received.

4.1.1 Annuity

Hu and Scott (2007) consider the discounted net value of an annuity from an individual's perspective at each time k as

$$X_c(k) = \sum_{j=0}^{k} v(0,j)c(j),$$

where $c(j)$ is the payoff of a single annuity, $v(0,k)$ is the deterministic discount factor from time k to time 0, and $X_c(k)$ the total discounted payoff the annuity holder obtains in case of death between k and $k+1$. As the initial wealth level is used as reference level, for a smaller value k, the investment in the retirement product usually leads to a loss. The value $X_c(k)$ then occurs with the subjective probability $\pi_k := \bar{s}_x(k) \cdot (1 - \bar{s}_{x+k}(1))$, the probability for an individual to die between k and $k+1$ from this individual's perspective. The CPT level of the annuity is then given by

$$\mathrm{CPT}_c = \sum_{k=0}^{\omega-x} \pi_k \cdot V(X_c(k)). \tag{5}$$

Note that, due to the deterministic payments of $c(j)$ and thus $V(X_c(k))$ being deterministic as well, no simulation is needed to compute the CPT value of the annuity.

4.1.2 Tontine

Taking the same approach as for the annuity, the (random) net value of a tontine at time k (given alive) can be expressed as

$$X_d(k) = \sum_{j=0}^{k} v(0,j)\frac{nd(j)}{N(j)}.$$

We assume that individuals estimates their own survival probabilities independently of the survival probabilities of their peers, allowing us to apply different probability distortions to both. The overall CPT value is then obtained by the following expression:[3]

$$\begin{aligned} \mathrm{CPT}_d &= \sum_{k=0}^{\omega-x} \pi_k \cdot \mathbb{E}\left[V(X_d(k)) \mid T_x > k\right] \\ &= \sum_{k=0}^{\omega-x} \pi_k \cdot \mathbb{E}\left[V\left(\sum_{j=0}^{k} v(0,j)\frac{nd(j)}{N(j)}\right) \;\middle|\; T_x > k\right], \end{aligned} \tag{6}$$

where the expectation again relies on subjective probabilities. Unlike annuities, the tontine payments depend on the realized number of the survivors at each time $k = 0,1,\ldots,\omega-x$. In order to compute the CPT value, we therefore need to rely on simulation techniques.

4.1.3 The main corollary

Under the preferences specified above and noting that they are continuous and increasing in consumption, we can immediately conclude the following.

Corollary 4.1 (Cumulative prospect theory). *Assume condition* (4). *Then, for any combination of a constant annuity and a natural tontine (except full annuitization), there exists a pool size* n_0 *such that an individual with CPT preferences prefers the combination to an equally priced constant annuity and an equally priced natural tontine to the combination, if the pool size in the tontine is at least* n_0.

[3] The CPT value $V(X_d(k))$ is only defined for a living individual. Using the property $\mathbb{E}[X \mid B] = \frac{\mathbb{E}[X\mathbb{1}_B]}{P(B)}$ for any set B occurring with positive probability and random variable X, we arrive at (6).

Proof: See Appendix A.3.

4.1.4 Numerical analysis

In a standard CPT framework, it is typically assumed that individuals underestimate probabilities close to 1 and overestimate probabilities close to 0. Furthermore, real-world probabilities p are typically transformed using a weighting function w (cf. Tversky and Kahneman (1992)). In the following, we will follow this approach and start by precising the computation of the CPT values of the annuity and tontine:

- For the annuity, it holds $X_c(0) < X_c(1) < X_c(2) < \dots$, given $c(j) > 0$ for all $j = 0, 1, \dots$. Therefore, for a probability weighting function w, we define

$$\begin{aligned}
&p_k := s_x(k) \cdot (1 - s_{x+k}(1)), \;\; k \in \{0, 1, \dots, \omega - x\} \\
&\pi_0 := w(p_0) \\
&\pi_k := w(p_0 + \dots + p_k) - w(p_0 + \dots + p_{k-1}), \;\; k \in \{1, 2, \dots, \omega - x\}.
\end{aligned}$$

  With this representation, we can directly compute the CPT value of the annuity (5).

- For the tontine, we use the fact that $(N(k) \mid T_x > k) \sim \mathrm{Bin}(n-1, s_x(k))$, given that the future lifetimes of the individuals are independent. From simulating $N(j)$, we obtain M different outcomes of $X_d(k)$, which we denote by $X_d^{(1)}(k) \leq X_d^{(2)}(k) \leq \dots \leq X_d^{(M)}(k)$. These outcomes occur with probabilities $P_k^{(1)}, P_k^{(2)}, \dots, P_k^{(M)}$. Define $\Pi_k^{(1)} := w(P_k^{(1)})$ and

$$\Pi_k^{(j)} := w(P_k^{(1)} + \dots + P_k^{(j)}) - w(P_k^{(1)} + \dots + P_k^{(j-1)}), \;\; j = 2, \dots, M.$$

  Thus, we can compute the CPT level of the tontine (6) as

$$\mathrm{CPT}_d = \sum_{k=0}^{\omega-x} \pi_k \cdot \sum_{j=1}^{M} \Pi_k^{(j)} \cdot V\left(X_d^{(j)}(k)\right).$$

Following Tversky and Kahneman (1992), we consider the value function below:

$$V(X) = \begin{cases} (X - \Gamma)^\beta, & X \geq \Gamma, \\ -\lambda \cdot (\Gamma - X)^\beta, & X < \Gamma, \end{cases} \tag{7}$$

where X is a risky prospect to be evaluated, $\lambda > 1$ the loss aversion parameter and $\beta \in (0,1)$ controls the sensitivity towards gains and losses. Following Tversky and Kahneman (1992) further, we consider the probability weighting function

$$w(p) = \frac{p^\nu}{(p^\nu + (1-p)^\nu)^{1/\nu}}, \quad \nu \in (0.28, 1], \tag{8}$$

but refrain from distinguishing between gains and losses, an assumption frequently made in the literature (cf. e.g. Ruß and Schelling (2018)). The lower bound for ν ensures that the probability weighting function is strictly increasing in p. Note that $\nu = 1$ delivers $w(p) = p$. Note that this probability weighting function leads individuals to overestimate probabilities close to zero and underestimate probabilities close to 1. In our context, this means that the probability that an individual's peers will survive extremely long will be overestimated, whereas the probability that they die at early ages will be overestimated as well. In particular, this violates (4).

As a mortality model, we consider the Gompertz mortality model which is parameterized by the modal age at death m and the dispersion coefficient b (equation (3) and Gompertz (1825)). Furthermore, we rely on the parameters summarized in Table 2.

Initial wealth $W_0 = 100$	Initial age $x = 65$	Risk-free interest (p.a.) $i = 0.01$
Loss aversion $\lambda = 2.25$	Sensitivity to gains/losses $\beta = 0.88$	Probability weighting $w(p) = (8)$, $\nu = 0.65$
Gompertz law ($\mathbb{P}$) $m = 88.721$, $b = 10$	Gompertz law ($\mathbb{Q}$) $m^{\mathbb{Q}} = 89.885$, $b^{\mathbb{Q}} = 10$	Maximum age $\omega = 120$

Table 2: Base case parameters.

Below, we provide some justification for these parameters:

- For the probability weighting function, we use the parameters $\nu = 0.65$ and $\nu = 1$ following, for example, Ruß and Schelling (2018). Note that this is the mean of the values 0.61 and 0.69 used by Tversky and Kahneman (1992).
- Following Tversky and Kahneman (1992), we set $\beta = 0.88$ and $\lambda = 2.25$.
- For simplicity, we assume a constant annual interest rate. Therefore, we obtain $v(0,k) = \left(\frac{1}{1+i}\right)^k =: v^k$ as the k-year discount factor. We choose a fairly low value to conform with the current situation in many countries.

- The Gompertz parameters b and m are chosen as in Milevsky and Salisbury (2015). They result in a 5% probability for a 65-year old to reach age 100.
- To achieve a prudent risk-neutral measure $\mathbb{Q}$, we follow the introductory numerical example described in Section 2. To be precise, we choose $m^{\mathbb{Q}}$ such that the proportional risk loading of a constant annuity is 4%, close to a risk margin determined for annuities in Chen et al. (2019). We obtain $m^{\mathbb{Q}} = 89.885$. For a natural tontine, we get a proportional risk loading of approximately 0.0056%. Note that the subjective survival probabilities are determined by the choice of ν, which results in both under- and overestimations of some best-estimate and risk neutral survival probabilities.
- Under the Gompertz parameters specified, the probabilities to reach age 120 are given by $s_{65}(55) = 1.34 \cdot 10^{-10}$ and $s_{65}^{\mathbb{Q}}(55) = 1.63 \cdot 10^{-9}$. which is sufficiently close to zero to assume a maximum age of 120.

To compare different retirement products, we rely on the certainty equivalent, the single deterministic payment exceeding the initial wealth level which delivers the same CPT value as some retirement product. To be precise, it is defined as $V(\mathrm{CE}_j + W_0) = \mathrm{CPT}_j$ for $j \in \{\text{annuity}, \text{tontine}\}$. A positive certainty equivalent (an addition to the initial wealth) speaks for the investment in the considered retirement product, while a negative one (a subtraction from the initial wealth) means that holding on to the initial wealth is preferred to buying a retirement plan.

In Table 3, we show the certainty equivalents of annuities and tontines with and without risk loadings in dependence of the pool size n. The pool sizes are chosen within the range of pool sizes considered in Qiao and Sherris (2013) who recommend a pool size of at least 1000. Although assumption (4) is not necessarily fulfilled in this example (since subjective probabilities may lead to over-and underestimations), we observe that tontines are preferred to annuities under all parameter combinations except for the case with no risk loadings and no subjective probabilities. Hence, these results numerically confirm that the result of Theorem 4.1 may go beyond assumption (4) (in particular, the results with subjective probabilities and no risk loadings). However, surprisingly, we observe that subjective probabilities lead individuals to prefer tontines with smaller pool sizes. In other words, in this case, individuals seek a volatile rather than a smooth retirement income. This finding is particularly new to the literature analyzing tontines under expected utility (Milevsky and Salisbury (2015) and Chen et al. (2020b)). Under subjective probabilities, the minimum pool size (which also deliver the highest CE) is

	$\lambda = 1$	$\lambda = 2.25$
Risk loadings		
$n = 100$		
$\nu = 0.65$	$(1.06, 6.63)$	$(-9.06, -6.15)$
$\nu = 1$	$(-2.32, 0.08)$	$(-9.39, -7.73)$
$n = 500$		
$\nu = 0.65$	$(1.06, 6.14)$	$(-9.06, -6.39)$
$\nu = 1$	$(-2.32, 0.117)$	$(-9.39, -7.685)$
$n = 1000$		
$\nu = 0.65$	$(1.06, 6.01)$	$(-9.06, -6.45)$
$\nu = 1$	$(-2.32, 0.121)$	$(-9.39, -7.680)$
No risk loadings		
$n = 100$		
$\nu = 0.65$	$(4.03, 5.06)$	$(-7.11, -6.58)$
$\nu = 1$	$(0.228, 0.205)$	$(-7.219, -7.252)$
$n = 500$		
$\nu = 0.65$	$(4.03, 4.50)$	$(-7.11, -6.87)$
$\nu = 1$	$(0.228, 0.222)$	$(-7.219, -7.229)$
$n = 1000$		
$\nu = 0.65$	$(4.03, 4.36)$	$(-7.11, -6.94)$
$\nu = 1$	$(0.228, 0.224)$	$(-7.219, -7.224)$

Table 3: Certainty equivalents of the annuity and the tontine. We rely on the base case parameters introduced in Table 2. In case of no risk loadings, $\mathbb{Q}$ equals $\mathbb{P}$.

therefore 2. With risk loadings and no subjective probabilities, CEs are increasing in the pool size. Here, we find again 2 to be the minimum pool size required for the tontine to be preferred to the annuity. Our results are consistent with Chen et al. (2020a) who also find 2 or 3 to be the minimum pool size for an agent subject to misspecified probabilities to prefer tontines to annuities.

Finally, note that Hu and Scott (2007) find loss aversion to be the main driving factor behind the unattractiveness of annuities. The negative certainty equivalents mostly obtained under loss aversion in Table 3 are consistent with this result. Nevertheless, we observe in Table 3 that it is possible that a tontine delivers a positive certainty equivalent while that of the annuity is negative.

4.2 Generalized expected utility theory

To evaluate different retirement plans under EUT, we consider generalized life-cycle utility preferences allowing for temporal risk aversion (see Bommier (2006) and Bommier et al. (2011)), i.e. agents evaluate the expected discounted lifetime utility

$$\mathbb{E}\left[\Phi\left(\sum_{k=0}^{\omega-x}\mathbb{1}_{\{T_x>k\}}\rho(0,k)u\left(C(k)\right)\right)\right],$$

where u and Φ are increasing and continuous functions, $\rho(0,k)$ is a subjective discount factor from time 0 to time k and $C(k)$ is a (possibly stochastic) consumption process. The case with a linear Φ results in the traditional life cycle utility preferences as used e.g. in Yaari (1965). A concave (convex) function Φ means that individuals are risk averse (loving) with respect to the length of life. Note that the typical assumption in the literature is for u and Φ to be concave, but we only require these functions to be increasing and continuous in order to transfer Theorem 3.1 to the above preferences. In the above expectation, similar to CPT, we assume that agents may be subject to a misspecification of their own and others' survival probabilities (which do not necessarily need to coincide).

Similar to the CPT valuation, payoffs of retirement plans need to satisfy the budget constraint that their initial value (determined under a risk-neutral measure) shall not exceed the initial wealth W_0.

4.2.1 The main corollary

Corollary 4.2 states that risk-averse, risk-neutral and risk-loving agents prefer a natural tontine to a constant annuity under assumption (4) and a sufficiently large pool size.

Corollary 4.2 (**Generalized expected utility**). *Assume condition* (4)*. Then, for any combination of a constant annuity and a natural tontine (except full annuitization), there exists a pool size* n_0 *such that an individual with EUT preferences prefers the combination to an equally priced constant annuity and an equally priced natural tontine to the combination, if the pool size in the tontine is at least* n_0*.*

Proof: See Appendix A.4.

4.2.2 Numerical analysis

In this subsection, we want to study EUT numerically. We fix the utility preferences as follows:

- Utility with respect to consumption is specified as a function of the constant relative risk aversion (CRRA) type, i.e.

$$u(y) = \frac{y^{1-\gamma}}{1-\gamma},$$

  where $\gamma \neq 1$ is the constant degree of relative risk aversion. We choose a frequently used value of $\gamma = 3$.

- Temporal risk aversion is captured by an exponential utility function of the form

$$\Phi(y) = -\frac{1}{\theta} e^{-\theta y}.$$

  where we choose $\theta = 0.035$ in a similar way as Bommier et al. (2011).

- For the subjective discount factor, we assume that it equals the risk-free interest rate. For such time preferences, a constant annuity is optimal under utility preferences with temporal risk neutrality (cf. Yaari (1965)).

The parameters are summarized in Table 4. In addition to these, we rely on the parameters in Table 2.

Temporal risk aversion	Risk aversion (consumption)	Subjective discount factor
$\theta = 0.035$	$\gamma = 3$	$i = 0.01$

Table 4: Base case parameters (generalized expected utility).

In this section, we compare retirement plans by certainty equivalent perpetuities, providing constant payments till the maximum age (regardless of whether alive or dead):

$$\Phi\left(\sum_{k=0}^{\omega-x} \rho(0,k) u(\mathrm{CE})\right) = \mathbb{E}\left[\Phi\left(\sum_{k=0}^{\omega-x} \mathbb{1}_{\{T_x>k\}} \rho(0,k) u\left(C(k)\right)\right)\right]$$

Solving this for CE delivers

$$\mathrm{CE} = \left(\frac{1-\gamma}{\sum_{k=0}^{\omega-x} \rho(0,k)} \Phi^{-1}\left(\mathbb{E}\left[\Phi\left(\sum_{k=0}^{\omega-x} \mathbb{1}_{\{T_x>k\}} \rho(0,k) u\left(C(k)\right)\right)\right]\right)\right)^{\frac{1}{1-\gamma}}.$$

In order to capture the underestimation of one's own and peers' survival probabilities required by (4), we vary the modal age at death m, because survival probabilities under the Gompertz mortality law are increasing in this variable. First, note that the base case delivers an approximate remaining lifetime of 20.70 years. Fixing $\bar{m} = \hat{m} = 82$, we obtain an underestimation of the average life expectancy by approximately 5.13 years, which falls into the range of the male and female values provided in O'Brien et al. (2005). We provide the resulting certainty equivalents in Table 5. We observe that, once

	$\gamma = 0.5$	$\gamma = 3$
Risk loadings		
$n = 100$		
$\bar{m} = \hat{m} = 84$	$(0.395, 0.418)$	$(8.40, 9.56)$
$m = \bar{m} = \hat{m}$	$(0.5464, 0.5525)$	$(7.71, 7.68)$
$n = 500$		
$\bar{m} = \hat{m} = 84$	$(0.394, 0.417)$	$(8.40, 9.60)$
$m = \bar{m} = \hat{m}$	$(0.5448, 0.5511)$	$(7.70, 7.94)$
$n = 1000$		
$\bar{m} = \hat{m} = 84$	$(0.394, 0.417)$	$(8.40, 9.60)$
$m = \bar{m} = \hat{m}$	$(0.5451, 0.5514)$	$(7.70, 7.95)$
No risk loadings		
$n = 100$		
$\bar{m} = \hat{m} = 84$	$(0.405, 0.424)$	$(8.74, 9.77)$
$m = \bar{m} = \hat{m}$	$(0.5590, 0.5587)$	$(8.01, 7.33)$
$n = 500$		
$\bar{m} = \hat{m} = 84$	$(0.405, 0.425)$	$(8.73, 9.81)$
$m = \bar{m} = \hat{m}$	$(0.56041, 0.56035)$	$(8.01, 7.86)$
$n = 1000$		
$\bar{m} = \hat{m} = 84$	$(0.406, 0.426)$	$(8.73, 9.82)$
$m = \bar{m} = \hat{m}$	$(0.55929, 0.55926)$	$(8.01, 7.95)$

Table 5: Certainty equivalents of the annuity and the tontine. We rely on the base case parameters introduced in Table 2 and 4. In case of no risk loadings, $\mathbb{Q}$ equals $\mathbb{P}$.

condition (4) is fulfilled, the tontine is preferred to the annuity regardless of the pool size (considered in the table), except for the case with risk loadings, no subjective mortality and a risk aversion of 3. This case requires a pool size larger than 100 for the tontine to be preferred to the annuity. We find a pool size of 115 to be sufficient. Only with no risk loadings and no subjective mortality beliefs the constant annuity is preferred to the natural tontine. In Table 6, we summarize the minimum pool sizes for the tontine to be preferred. The case with risk loadings, no subjective mortality and a risk aversion of 3 requires a particularly large pool size in this example, because the difference in survival

	$\gamma = 0.5$	$\gamma = 3$
Risk loadings		
$\bar{m} = \hat{m} = 84$	$n_0 = 2$	$n_0 = 5$
$m = \bar{m} = \hat{m}$	$n_0 = 2$	$n_0 = 115$
No risk loadings		
$\bar{m} = \hat{m} = 84$	$n_0 = 2$	$n_0 = 8$

Table 6: Certainty equivalents of the annuity and the tontine. We rely on the base case parameters introduced in Table 2 and 4. In case of no risk loadings, $\mathbb{Q}$ equals $\mathbb{P}$.

probabilities resulting from only the risk loadings is less pronounced than in the presence of subjective mortality and due to the relatively higher degree of relative risk aversion. These circumstances lead an individual to prefer an annuity over tontines with a smaller pool size.

5 Conclusion

We show that retirement benefits of a properly designed tontine with an infinite pool size dominate the benefits of an equally priced annuity. The main assumption for this finding is the presence of risk loadings for retirement benefits or subjective mortality beliefs underestimating survival probabilities compared to the insurer. This result is a generalization of the findings of previous articles in different normative and descriptive model setups: Particularly under realistic risk loadings, at least partial tontinization combined with partial annuitization is preferred to full annuitization if the tontine pool is large enough. Therefore, this article raises a so-called tontine puzzle, referring to the discrepancy between the theoretically optimal demand for tontines and their actual development in practice.

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A Proofs

A.1 Proof of Theorem 3.1

The constant annuity payoff is given by $c = W_0 / \sum_{k=0}^{\omega-x} v(0,k) s_x^{\mathbb{Q}}(k)$. The tontine payoff is given by $d(k) = \mathbb{E}_{\mathbb{Q}}\left[\mathbb{1}_{\{T_x > k\}}\right] d_0 = s_x^{\mathbb{Q}}(k) d_0$, where

$$d_0 = \frac{W_0}{\sum_{k=0}^{\omega-x} v(0,k) s_x^{\mathbb{Q}}(k)\left(1 - (1 - s_x^{\mathbb{Q}}(k))^n\right)} \underset{n\to\infty}{\to} c.$$

Note that the tontine payoff therefore converges as follows from a living agent's perspective:

$$\frac{nd(j)}{N(j)} = \frac{d_0 s_x^{\mathbb{Q}}(j)}{N(j)/n} \underset{n\to\infty}{\to} \frac{c s_x^{\mathbb{Q}}(j)}{\hat{s}_x(j)} > c \text{ a.s. for all } j \in \{0, 1, \ldots, \omega - x\}.$$

□

A.2 Proof of Corollary 3.3

Let $U(\cdot)$ be an increasing and continuous real-valued function and assume that an individual invests ηW_0 in the annuity and $(1-\eta)W_0$ in the tontine, where $\eta \in (0,1)$.

Then, by continuity, we have

$$\lim_{n\to\infty} U\left(\left\{\frac{nd(j)}{N(j)}\right\}_{j\in\{0,1,\ldots,\omega-x\}}\right) = U\left(\left\{\frac{cs_x^{\mathbb{Q}}(j)}{\hat{s}_x(j)}\right\}_{j\in\{0,1,\ldots,\omega-x\}}\right),$$

and

$$\begin{aligned}&\lim_{n\to\infty} U\left(\left\{(1-\eta)\frac{nd(j)}{N(j)}+\eta c\right\}_{j\in\{0,1,\ldots,\omega-x\}}\right)\\ &= U\left(\left\{(1-\eta)\frac{cs_x^{\mathbb{Q}}(j)}{\hat{s}_x(j)}+\eta c\right\}_{j\in\{0,1,\ldots,\omega-x\}}\right).\end{aligned}$$

Due to the monotonicity of U, we then have

$$\begin{aligned}U\left(\left\{\frac{cs_x^{\mathbb{Q}}(j)}{\hat{s}_x(j)}\right\}_{j\in\{0,1,\ldots,\omega-x\}}\right) &> U\left(\left\{(1-\eta)\frac{cs_x^{\mathbb{Q}}(j)}{\hat{s}_x(j)}+\eta c\right\}_{j\in\{0,1,\ldots,\omega-x\}}\right)\\ &> U(\{c\}_{j\in\{0,1,\ldots,\omega-x\}})\end{aligned}$$

because $\frac{cs_x^{\mathbb{Q}}(j)}{\hat{s}_x(j)} > c$ for all $j \in \{0,1,\ldots,\omega-x\}$. Due to basic properties of convergence (see, e.g., Schulz (2011)), there must therefore exist some pool size n_0 such that the utility of a finite tontine with pool size larger or equal to n_0 exceeds the utility of the combination and such that the utility of the combination exceeds the utility of the annuity.

□

A.3 Proof of Corollary 4.1

Note that, due to the continuity and monotonicity of V, cumulative prospect theory preferences are a special case of the preferences covered in Corollary 3.3. The CPT level of the tontine is given by

$$\begin{aligned}\mathrm{CPT}_d &= \sum_{k=0}^{\omega-x} \pi_k \cdot \mathbb{E}\left[V(X_d(k)) \mid T_x > k\right]\\ &= \sum_{k=0}^{\omega-x} \pi_k \cdot \mathbb{E}\left[V\left(\sum_{j=0}^{k} v(0,j)\frac{nd(j)}{N(j)}\right) \;\Big|\; T_x > k\right]\end{aligned}$$

$$\underset{n\to\infty}{\to} \sum_{k=0}^{\omega-x} \pi_k \cdot V\left(\sum_{j=0}^{k} v(0,j) c \frac{s_x^{\mathbb{Q}}(k)}{\hat{s}_x(k)}\right),$$

where we apply the continuity property in the convergence result. Due to the monotonicity of V, we then obtain

$$\sum_{k=0}^{\omega-x} \pi_k \cdot V\left(\sum_{j=0}^{k} v(0,j) c \frac{s_x^{\mathbb{Q}}(k)}{\hat{s}_x(k)}\right) > \sum_{k=0}^{\omega-x} \pi_k \cdot V\left(\sum_{j=0}^{k} v(0,j) c\right) = \mathrm{CPT}_c$$

which proves the claim by basic properties of convergence. Now let again $\eta \in (0,1)$ denote the fraction of wealth invested in the annuity. Analogously, we can show that

$$\begin{aligned}
&\sum_{k=0}^{\omega-x} \pi_k \cdot \mathbb{E}\left[V\left(\sum_{j=0}^{k} v(0,j)\left((1-\eta)\frac{nd(j)}{N(j)} + \eta c\right)\right) \;\middle|\; T_x > k\right] \\
&\underset{n\to\infty}{\to} \sum_{k=0}^{\omega-x} \pi_k \cdot V\left(\sum_{j=0}^{k} v(0,j)\left((1-\eta) c \frac{s_x^{\mathbb{Q}}(k)}{\hat{s}_x(k)} + \eta c\right)\right) \\
&\begin{cases} > \sum_{k=0}^{\omega-x} \pi_k \cdot V\left(\sum_{j=0}^{k} v(0,j) c\right) \\ < \sum_{k=0}^{\omega-x} \pi_k \cdot V\left(\sum_{j=0}^{k} v(0,j) c \frac{s_x^{\mathbb{Q}}(k)}{\hat{s}_x(k)}\right) \end{cases}.
\end{aligned}$$

□

A.4 Proof of Corollary 4.2

Due to the continuity of Φ and u, we get

$$\mathbb{E}\left[\Phi\left(\sum_{k=0}^{\omega-x} \mathbb{1}_{\{T_x>k\}} \rho(0,k) u\left(\frac{nd(k)}{N(k)}\right)\right)\right] \underset{n\to\infty}{\to} \mathbb{E}\left[\Phi\left(\sum_{k=0}^{\omega-x} \mathbb{1}_{\{T_x>k\}} \rho(0,k) u\left(c\frac{s_x^{\mathbb{Q}}(k)}{\hat{s}_x(k)}\right)\right)\right].$$

Due to the monotonicity of Φ and u, we then obtain

$$\mathbb{E}\left[\Phi\left(\sum_{k=0}^{\omega-x} \mathbb{1}_{\{T_x>k\}} \rho(0,k) u\left(c\frac{s_x^{\mathbb{Q}}(k)}{\hat{s}_x(k)}\right)\right)\right] > \mathbb{E}\left[\Phi\left(\sum_{k=0}^{\omega-x} \mathbb{1}_{\{T_x>k\}} \rho(0,k) u\left(c(k)\right)\right)\right].$$

Now let again $\eta \in (0,1)$ denote the fraction of wealth invested in the annuity. Then, analogously, we get

$$\begin{aligned}
&\mathbb{E}\left[\Phi\left(\sum_{k=0}^{\omega-x}\mathbb{1}_{\{T_x>k\}}\rho(0,k)u\left((1-\eta)\frac{nd(k)}{N(k)}+\eta c\right)\right)\right]\\
&\underset{n\to\infty}{\rightarrow}\ \mathbb{E}\left[\Phi\left(\sum_{k=0}^{\omega-x}\mathbb{1}_{\{T_x>k\}}\rho(0,k)u\left((1-\eta)c\frac{s_x^{\mathbb{Q}}(k)}{\hat{s}_x(k)}+\eta c\right)\right)\right]\\
&\begin{cases} > \mathbb{E}\left[\Phi\left(\sum_{k=0}^{\omega-x}\mathbb{1}_{\{T_x>k\}}\rho(0,k)u\left(c(k)\right)\right)\right]\\ < \mathbb{E}\left[\Phi\left(\sum_{k=0}^{\omega-x}\mathbb{1}_{\{T_x>k\}}\rho(0,k)u\left(c\frac{s_x^{\mathbb{Q}}(k)}{\hat{s}_x(k)}\right)\right)\right]. \end{cases}
\end{aligned}$$

□